* Rotational Dynamics *

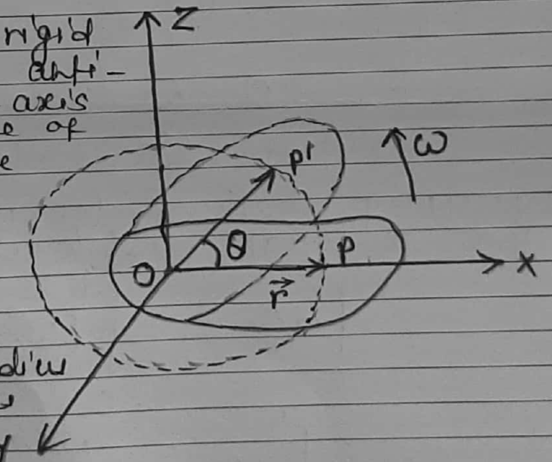
Rigid body:- A body is said to be rigid if it does not undergo any change in its size and shape, however large the external force may be acting on it.

A rigid body is one whose constituent particles retain their relative positions even when they move under the action of an external force.

A rigid body cannot be deformed. If the body undergoes some displacement, every particle in it suffers the same displacement. If the body rotates through a certain angle, every particle of it rotates through the same angle about the axis of rotation.

Rotational motion:- A body is said to possess rotational motion if all its particles move along circles in parallel planes and centre of these circles lie on a fixed line perpendicular to the parallel planes.

** Consider a rigid body being rotated anti-clockwise about z-axis of an inertial frame of reference. Let P be any particle of a body and $\vec{r}$ is the position vector. As body rotates, the particle P moves along a circle of radius r whose centre lies on the axis of rotation.



The radius vector $\vec{r}$ sweeps out an angle θ in certain time t. Similarly, all other particles of rigid body move along circles with their centres on z-axis and their radius vector sweep the same angle θ in time t. So all the particles have the same angular velocity which is also the angular velocity of the body. **

Equations of rotational motion

Consider a rigid body rotating about fixed axis with constant angular acceleration. Let dω is the small change in angular velocity in time dt, then angular acceleration of a body is

$$\alpha = \frac{d\omega}{dt}$$

or $d\omega = \alpha dt$

Integrating both sides within the limits of time and angular velocity, we get

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt \qquad \left[\because \text{At } t=0,\ \omega=\omega_0 \ \& \ t=t,\ \omega=\omega\right]$$

or, $\left[\omega\right]_{\omega_0}^{\omega} = \alpha\left[t\right]_0^t$

or, $\omega - \omega_0 = \alpha(t-0)$

or, $\boxed{\omega = \omega_0 + \alpha t}$ ——— (i)

Also, let $d\theta$ is the small angular displacement covered in time dt, then angular velocity of a rigid body is

$$\omega = \frac{d\theta}{dt}$$

or, $d\theta = \omega dt$

or, $d\theta = (\omega_0 + \alpha t)dt$

Integrating both sides within the limits of time and angular displacement, we get

$$\int_0^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)dt$$

or, $\int_0^{\theta} d\theta = \int_0^t \omega_0 dt + \alpha\int_0^t t dt$

or, $\left[\theta\right]_0^{\theta} = \omega_0\left[t\right]_0^t + \alpha\left[\frac{t^2}{2}\right]_0^t$

or, $\theta - 0 = \omega_0(t-0) + \frac{\alpha}{2}(t^2-0)$

$\therefore \boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$ ——— (ii)

Also, the angular acceleration may be expressed as

$$\alpha = \frac{d\omega}{dt}$$

or, $\alpha = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt}$

$= \frac{d\omega}{d\theta} \times \omega$

or, $\omega\, d\omega = \alpha\, d\theta$

Integrating both sides we get

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_0^{\theta} \alpha\, d\theta$$

or, $\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha\left[\theta\right]_0^{\theta}$

or, $\frac{\omega^2}{2} - \frac{\omega_0^2}{2} = \alpha(\theta - 0)$

or, $\omega^2 - \omega_0^2 = 2\alpha\theta$

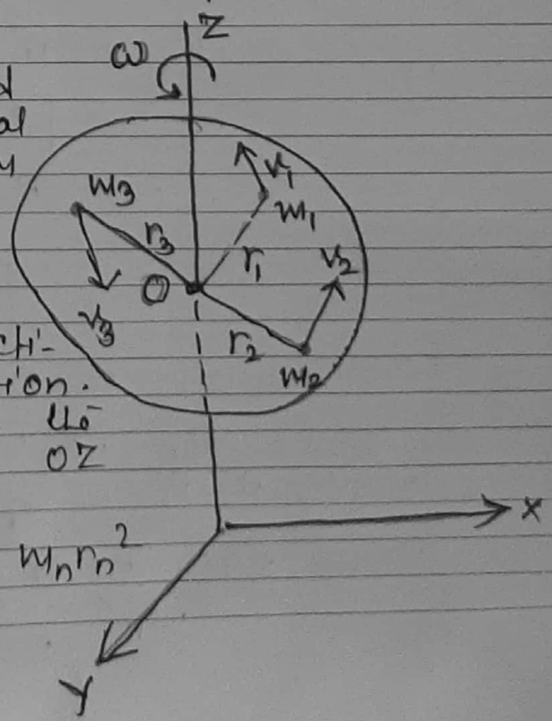
$$\therefore \boxed{\omega^2 = \omega_0^2 + 2\alpha\theta} \quad \text{——— (iii)}$$

Equations (i), (ii) and (iii) are the equations of motion for a body in rotational motion under constant angular acceleration.

Moment of inertia of a rigid body

The moment of inertia of a rigid body about a fixed axis is defined as the sum of the products of masses of particles constituting the body and the square of their respective distances from the axis of rotation.

Consider a rigid body rotating about vertical axis through O with uniform angular velocity ω. Let the body consists of n particles of masses $m_1, m_2, \ldots, m_n$ situated at distances $r_1, r_2, \ldots, r_n$ respectively from the axis of rotation. The moment of inertia of the rigid body about an axis OZ is



$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2$$

$$\therefore I = \sum_{i=1}^{i=n} m_i r_i^2$$

The SI unit of moment of inertia is $kg\,m^2$ and CGS unit is $gm\,cm^2$.
The dimensional formula of moment of inertia is $[M^1L^2T^0]$.

Physical Significance of M.I.

The moment of inertia of a body plays the same role in the rotational motion as the mass plays in linear motion. That is why moment of inertia is called rotational analogue of mass in linear motion.

Note:- Moment of inertia of body depends on the following factors –

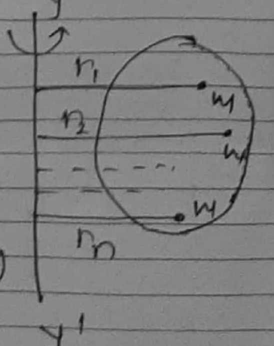
(i) Mass of the body.
(ii) Size and shape of the body.
(iii) Distribution of mass about the axis of rotation.
(iv) Position and orientation of the axis of rotation with respect to the body.

Radius of gyration

The radius of gyration of a body about its axis of rotation may be defined as the distance from the axis of rotation at which, if the whole mass of the body were concentrated, its moment of inertia about the given axis would be the same as with the actual distribution of mass. It is denoted by K.

Consider a rigid body consists of n particles each of mass m, situated at distances $r_1, r_2, \ldots r_n$ from the axis of rotation YY'. The moment of inertia of the body about the axis YY' is

$$I = mr_1^2 + mr_2^2 + \cdots + mr_n^2$$
$$= m(r_1^2 + r_2^2 + \cdots + r_n^2)$$
$$= m \times n \left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}\right)$$
$$= M\left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}\right) \quad \text{—— (I)}$$



Where, $M = m \times n$ = total mass of the body.

If K is the radius of gyration about the axis YY', then

$$I = MK^2 \quad \text{—— (II)}$$

From eqs. (I) & (II), we get

$$MK^2 = M\left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}\right)$$

$$\therefore \boxed{K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}}$$

Hence the radius of gyration of a body about an axis of rotation may also be defined as the root mean square distance of its particles from the axis of rotation.

Note:- Radius of gyration of a body depends on the following factors -

(i) Position and direction of the axis of rotation.

(ii) Distribution of mass about the axis of rotation.

Relation between rotational kinetic energy and moment of inertia

Consider a rigid body rotating about an axis OZ with uniform angular velocity ω. Let rigid body consists of n particles of masses $m_1, m_2, \ldots, m_n$ situated at distances $r_1, r_2, \ldots, r_n$ from the axis of rotation. As the angular velocity of all the particles is same, so their linear velocities are

$$v_1 = r_1\omega, \quad v_2 = r_2\omega \quad \ldots\ldots, \quad v_n = r_n\omega$$

Thus, the total kinetic energy of rotation of the body about the axis OZ is

$$K.E. = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots + \frac{1}{2}m_nv_n^2$$

$$= \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega)^2 + \cdots + \frac{1}{2}m_n(r_n\omega)^2$$

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \cdots + \frac{1}{2}m_nr_n^2\omega^2$$

$$= \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + \cdots + m_nr_n^2)\,\omega^2$$

$$= \frac{1}{2}\left(\sum_{i=1}^{i=n} m_ir_i^2\right)\omega^2$$

$$= \frac{1}{2}I\omega^2$$

$\left[\because I = \sum_{i=1}^{i=n} m_ir_i^2,\right.$ the M.I. of body about the axis of rotation $\left.\right]$

$$\therefore \boxed{K.E_{rot.} = \frac{1}{2}I\omega^2}$$

If $\omega = 1$ rad/s, then

$$K.E_{rot.} = \frac{1}{2} I$$

$$\therefore \boxed{I = 2 \times K.E_{rot.}}$$

Hence, the moment of inertia of a rigid body about the axis of rotation is numerically equal to twice the rotational kinetic energy of the body when it is rotating with unit angular velocity about that axis.

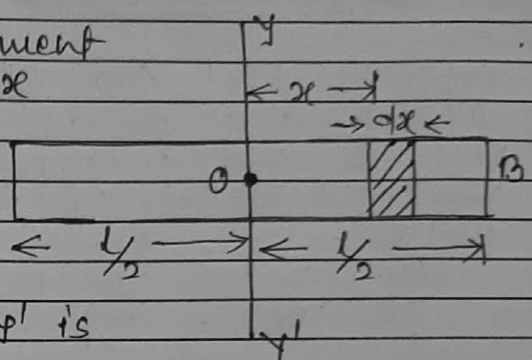
Moment of inertia of thin uniform rod:—

(i) About an axis through its centre of mass and perpendicular to its length:—

Consider a thin uniform rod AB of length l and mass M, free to rotate about an axis YY' through its centre of mass O and perpendicular to its length.

$\therefore$ Mass per unit length of rod $= M/l$

Consider a small mass element of length dx at a distance x from O, then mass of an element



$$= \frac{M}{l}dx$$

Moment of inertia of the element about an axis YY' is

$$dI = (\text{Mass of element}) \times (\text{distance})^2$$

$$= \left(\frac{M}{l}dx\right) \times x^2 \qquad \text{——— (i)}$$

Therefore, the moment of inertia of the whole rod about the axis can be obtained by integrating equation (i) between the limits $x = -l/2$ to $x = +l/2$. Then,

$$I = \int dI = \int_{-l/2}^{+l/2} \frac{M}{l}x^2 dx$$

$$= \frac{M}{l}\int_{-l/2}^{+l/2} x^2 dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{+l/2}$$

$$= \frac{M}{3l}\left[(l/2)^3 - (-l/2)^3\right]$$

$$= \frac{M}{3l}\left[\frac{l^3}{8} + \frac{l^3}{8}\right]$$

$$= \frac{M}{3l} \times \frac{l^3}{4}$$

$$\therefore \boxed{I = \frac{Ml^2}{12}} \qquad \text{——— (ii)}$$

Note:- Let K be the radius of gyration of the rod about the axis YY', then

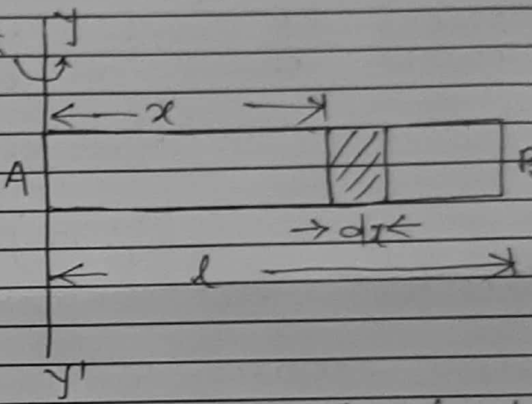
$$I = MK^2 = \frac{Ml^2}{12}$$

$$\therefore \boxed{K = \frac{l}{2\sqrt{3}}}$$

(ii) About an axis passing through one end and perpendicular to its length:-

Consider a thin uniform rod AB of length l and mass M, free to rotate and an axis YY' passing through its one end A and perpendicular to its length about which moment of inertia has to be ~~displace~~ determined.

Now, mass per unit length of rod $= M/l$



Consider a small mass element of length dx of the rod at a distance x from the end A, then the mass of an element

$$= \frac{M}{l}dx$$

The moment of inertia of an element about an axis YY' is

$$dI = (\text{Mass of element}) \times (\text{distance})^2$$

$$= \left(\frac{M}{l}dx\right)x^2 \quad \text{——— (i)}$$

Therefore, the moment of inertia of whole rod about an axis YY' can be obtained by integrating eq. (i) under the respective limits. Thus

$$I = \int dI = \int_0^l \frac{M}{l}x^2 dx$$

$$= \frac{M}{l}\int_0^l x^2 dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{M}{3l}\left[l^3 - 0\right]$$

$$\boxed{I = \frac{1}{3}Ml^2} \quad \text{——— (ii)}$$

Note:- Let K be the radius of gyration of rod about YY' then

$$I = MK^2 = \frac{1}{3}Ml^2 \quad \therefore \boxed{K = l/\sqrt{3}}$$

Relation between torque and moment of inertia

Consider a rigid body consists of n particles of masses $m_1, m_2, \ldots m_n$; apart at distances $r_1, r_2, \ldots r_n$ from the axis YY'. When a torque acts on a body capable of rotation about an axis, it produces an angular acceleration in the body and will same for all particles of the body. The linear acceleration of the particles depends on their distances $r_1, r_2 \ldots r_n$ from the axis.

Now, the force acting on the particles

$$F_1 = m_1 a_1 = m_1 r_1 \alpha, \quad F_2 = m_2 a_2 = m_2 r_2 \alpha \ldots F_n = m_n r_n \alpha$$

The torque acting on the rigid body is the sum of the moment of forces of the particles about the axis of rotation. So

$$\tau = \tau_1 + \tau_2 + \ldots + \tau_n$$

$$= m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + \ldots + m_n r_n^2 \alpha$$

$$[\because \tau_1 = F_1 r_1, \ \tau_2 = F_2 r_2 \ldots \tau_n = F_n r_n]$$

$$= [m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2]\alpha$$

$$= \left(\sum_{i=1}^{i=n} m_i r_i^2\right)\alpha$$

$$= I\alpha \qquad \left[\because I = \sum_{i=1}^{i=n} m_i r_i^2, \text{ Moment of inertia of a rigid body}\right]$$

$$\therefore \boxed{\tau = I\alpha}$$

If $\alpha = 1\ rad/s^2$ then

$$\boxed{\tau = I}$$

Hence, the moment of inertia of a rigid body about an axis of rotation is numerically equal to the external torque required to produce unit angular acceleration in the body about that axis.

Angular Momentum:-

The angular momentum of a particle rotating about an axis is defined as the moment of the linear momentum of the particle about that axis.

It is measured by the product of linear momentum and perpendicular distance of its line of action from the axis of rotation.

Therefore,

$L =$ Linear momentum × Perpendicular distance.

or, $\boxed{L = pr}$

or, $\boxed{L = mvr}$ — (i) $[\because p = mv]$

If ω be the angular velocity of a particle then $v = r\omega$. So

$$L = m(r\omega)r$$

or, $L = mr^2\omega$

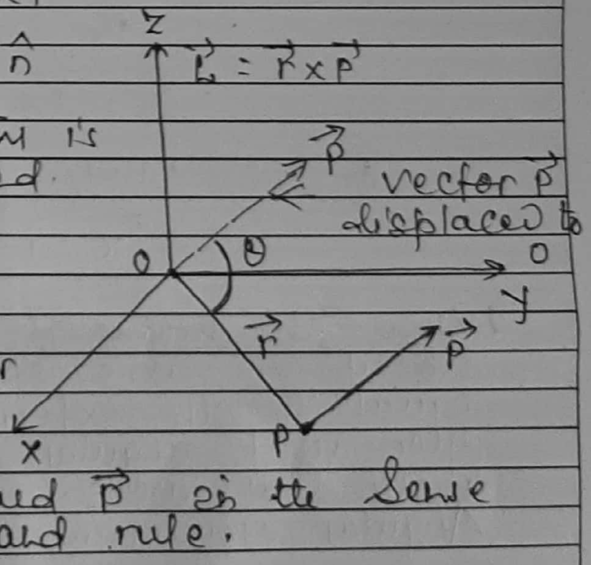
$\therefore \boxed{L = I\omega}$ ——— (ii)

Angular Momentum in vector form:—

Consider a particle P of mass 'm' rotating about an axis through O in X-Y plane. Suppose the particle has linear momentum $\vec{p}$ which makes angle θ with its position vector $\vec{OP} = \vec{r}$. The angular momentum $\vec{L}$ of the particle about the origin O is defined as the cross product of vectors $\vec{r}$ and $\vec{p}$. Thus

$$\vec{L} = \vec{r} \times \vec{p}$$

or, $\vec{L} = rp\sin\theta\,\hat{n}$



The angular momentum is a vector quantity and its magnitude is

$$L = rp\sin\theta$$

The direction of angular momentum $\vec{L}$ is perpendicular to the plane of vectors $\vec{r}$ and $\vec{p}$ in the sense given by the right hand rule.

If $\theta = 0°$ or $180°$, $\sin\theta = 0$

$\therefore \boxed{L = 0}$

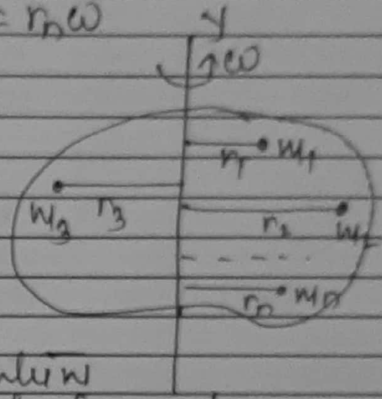
If $\theta = 90°$, $\sin 90° = 1$

$\therefore \boxed{L = rp}$ Maxm.

Relation between Angular momentum and Moment of inertia of a rigid body

Consider a rigid body rotating about fixed axis YY' with uniform angular velocity ω. Let a rigid body consists of n particles of masses $m_1, m_2 \ldots m_n$, situated at distance $r_1, r_2 \ldots r_n$ from the axis of rotation. The angular velocity ω of all the particles will be same but their linear velocities will be different and are given by

$$v_1 = r_1\omega, \quad v_2 = r_2\omega, \ldots \quad v_n = r_n\omega$$



The linear momentum of first particle,

$$P_1 = m_1 v_1.$$
$$= m_1 (r_1\omega)$$

Moment of linear momentum i.e, angular momentum of first particle

$$L_1 = P_1 r_1 = m_1 (r_1\omega) r_1$$
$$= m_1 r_1^2 \omega$$

The angular momentum of a rigid body about an axis is the sum of moments of linear momenta of all the particles about that axis. Thus

$$L = L_1 + L_2 + \cdots + L_n$$
$$= m_1 r_1^2\omega + m_2 r_2^2\omega + \cdots + m_n r_n^2 \omega$$
$$= (m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2)\omega$$
$$= \left(\sum_{i=1}^{i=n} m_i r_i^2\right)\omega$$
$$= I\omega$$

$$\therefore \boxed{L = I\omega} \qquad \left[\because I = \sum_{i=1}^{i=n} m_i r_i^2, \text{ moment of inertia of rigid body about YY'}\right]$$

If $\omega = 1$ rad/s, then

$$\boxed{L = I}$$

Hence, the moment of inertia of a rigid body about an axis is numerically equal to the angular momentum of the rigid body when rotating with unit angular velocity about that axis.

Principle of Conservation of angular momentum

Statement:- "If no external torque acts on a system, then the total angular momentum of the system remains conserved."

Mathematically,

$$I\omega = \text{Constant}.$$

Here, I is the moment of inertia of a body or system about given axis of rotation and ω is its angular velocity.

Proof:- Consider a body rotating about an axis with the angular velocity ω, whose moment of inertia about an axis is I, then the angular momentum of a body is

$$L = I\omega \quad \text{——— (i)}$$

Let $d\omega$ is the small change in angular velocity in time dt as external torque τ is applied then

$$\tau = I\alpha$$

$$= I\left(\frac{d\omega}{dt}\right)$$

$$= \frac{d(I\omega)}{dt} \qquad \left[\because I \text{ is constant about an axis}\right]$$

$$\boxed{\tau = \frac{dL}{dt}} \quad \text{——— (ii)}$$

If no external torque acting on a body, i.e., $\tau = 0$, then

$$0 = \frac{dL}{dt}$$

$$\text{or, } \frac{dL}{dt} = 0$$

$$\text{or, } dL = 0$$

On integrating, we get

$$L = \text{Constant}$$

$$\therefore \boxed{I\omega = \text{Constant}} \quad \text{——— (iii)}$$

This is the principle of conservation of angular momentum. It is also known as basic dynamic principle for rotation.

Note:- If I_1 and ω_1 are the initial values of M.I. and angular velocity of a system respectively and I_2 and ω_2 are their respective final values, then

$$\boxed{I_1\omega_1 = I_2\omega_2}$$